Hierarchical Polynomial Approximation

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Two-Level Polynomial Approximations

The problem: **After scaling, values of $f(0)$, $f(1)$, $f(2)$, ..., $f(T_I - 1)$ on a large interval I, of width T_I?**

- On I, f is approximated by a degree-d polynomial P, with an approximation error bounded by ϵ.
- I is split into sub-intervals J_0, J_1, J_2, ... of width T_J, where $J_k = J_{k-1} + T_J$.
- On each of the J_k, we are looking for an approximation Q_k of degree $\delta < d$.

$$P_k(X) = P(X + k \cdot T_J) = a_0(k) + a_1(k) \cdot X + a_2(k) \cdot \frac{X(X-1)}{2} + \cdots + a_d(k) \cdot \frac{X(X-1)(X-2)\cdots(X-d+1)}{d!}$$

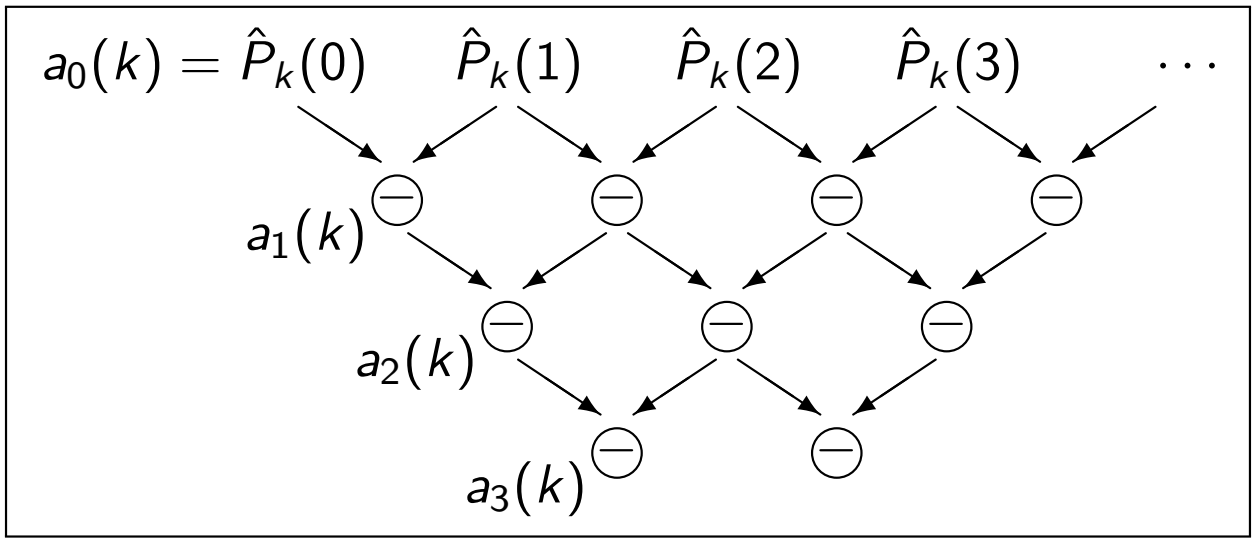
$$Q_k(X) = a_0(k) + a_1(k) \cdot X + a_2(k) \cdot \frac{X(X-1)}{2} + \cdots + a_\delta(k) \cdot \frac{X(X-1)(X-2)\cdots(X-\delta+1)}{\delta!} = \sum_{i=0}^{\delta} a_i(k) \cdot \binom{X}{i}$$

Computation of the Coefficients of $a_i(k)$

The problem: **Compute the values of $a_i(k)$.**

- For each $i \in [\![0, \delta]\!]$, $a_i(k)$ is a degree-$(d-i)$ polynomial function of k.
- We can compute the $a_i(k)$ from the $d-i$ consecutive values $P_k(0)$, $P_k(1)$, $P_k(2)$, ..., $P_k(d-i-1)$ for the first values of k; the following $a_i(k)$'s are computed with the *difference table method*.
- Let $P_k(u)$ be the "true" value, and $\hat{P}_k(u)$ be the "computed" value.

For $0 \leq k \leq d-i$:



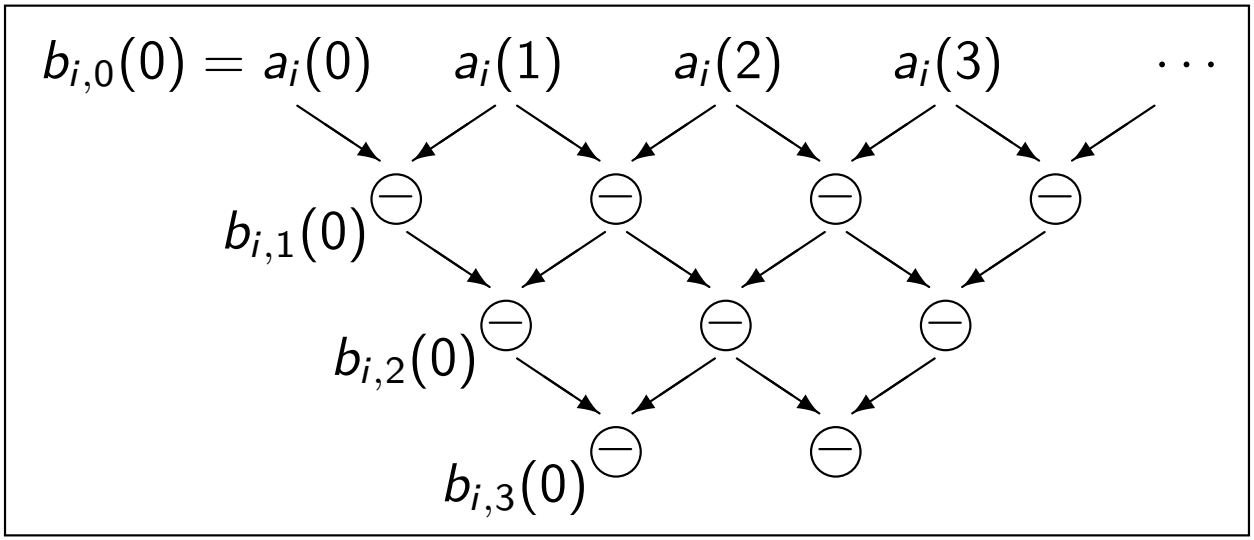
Let e be an upper bound on $\left|P_k(u) - \hat{P}_k(u)\right|$.

With exact subtractions, $a_i(k)$ will be known with an error $\leq 2^i e$.

Computation of the Coefficients of $a_i(k)$ [2]

Define values $b_{i,j}(k)$ as

$$a_i(k) = b_{i,0}(k) + b_{i,1}(k) \cdot k + b_{i,2}(k) \cdot \frac{k(k-1)}{2} + \cdots +$$
$$b_{i,d-i}(k) \cdot \frac{k(k-1)(k-2)\cdots(k-d+i+1)}{(d-i)!} = \sum_{j=0}^{d-i} b_{i,j}(k) \cdot \binom{k}{j}$$



The term $b_{i,j}(0)$ is known with an error $\leq 2^{i+j}e$.

The $a_i(k)$ will be computed with additions from these coefficients $b_{i,j}(0)\ldots$

Computation of Consecutive Values of a Polynomial

Let $Q(X+k) = \sum_{i=0}^{\delta} q_i(k) \cdot \binom{X}{i}$ of degree (at most) δ, where the $q_i(k)$'s are assumed to be the exact coefficients.

The problem: **Knowing the initial coefficients $q_i(0)$ of $Q(X)$, compute the consecutive values $Q(0)$, $Q(1)$, $Q(2)$, ...**

We have:
$$\begin{cases} q_0(k) &= q_0(k-1) &+ q_1(k-1) \\ q_1(k) &= q_1(k-1) &+ q_2(k-1) \\ & \vdots & \\ q_{\delta-1}(k) &= q_{\delta-1}(k-1) &+ q_\delta(k-1) \end{cases}$$

q_δ being a constant, and $Q(k) = q_0(k)$.

The coefficients $q_i(k)$ will be represented by $\hat{q}_i(k)$ with n_i bits after the fractional point, i.e. on n_i bits as we are interested in the values modulo 1, and an initial error of one ulp: $u_i = 2^{-n_i}$. Since $q_i(k)$ depends on $q_{i+1}(k-1)$, we assume that (n_i) is increasing: $n_i \leq n_{i+1}$. And $n_\delta = n_{\delta-1}$ for the constant coefficient q_δ (using more precision would be useless).

Computation of Consecutive Values of a Polynomial [2]

Formally: $n_i \leq n_{i+1}$, $u_i = 2^{-n_i}$, $\hat{q}_i(k) \in u_i\mathbb{Z}$ or $u_i\mathbb{Z}/\mathbb{Z}$, and $|\hat{q}_i(0) - q_i(0)| \leq u_i$.

Basic iteration:
$$\begin{cases} \hat{q}_0(k) = \hat{q}_0(k-1) + \circ(\hat{q}_1(k-1)) \\ \hat{q}_1(k) = \hat{q}_1(k-1) + \circ(\hat{q}_2(k-1)) \\ \vdots \\ \hat{q}_{\delta-2}(k) = \hat{q}_{\delta-2}(k-1) + \circ(\hat{q}_{\delta-1}(k-1)) \\ \hat{q}_{\delta-1}(k) = \hat{q}_{\delta-1}(k-1) + \hat{q}_\delta \end{cases}$$

where $|\hat{q}_\delta - q_\delta| \leq u_{\delta-1}$ and $\circ$ denotes the truncation of the value to the precision of the result (said otherwise, when doing an addition, the trailing bits of the more precise value are ignored).

Let $\epsilon_i(k)$ be a bound on the error on $q_i(k)$, i.e. $|\hat{q}_i(k) - q_i(k)| \leq \epsilon_i(k)$.
Initially, $\epsilon_i(0) = u_i$ for $0 \leq i \leq \delta - 1$.

We have: $\epsilon_i(k) = \epsilon_i(k-1) + \epsilon_{i+1}(k-1) + u_i$ for $0 \leq i \leq \delta - 1$,
with $\epsilon_\delta(0) = 0$ in order to satisfy $\epsilon_{\delta-1}(k) = \epsilon_{\delta-1}(k-1) + u_{\delta-1}$.

We can prove by induction that $\epsilon_i(k) = \sum_{j=i}^{\delta-1} u_j \cdot \binom{k+1}{j-i+1}$.

Computation of Consecutive Values of a Polynomial [3]

Proof by induction that $\epsilon_i(k) = \sum_{j=i}^{\delta-1} u_j \cdot \binom{k+1}{j-i+1}$.

This is true for $k = 0$ and for $i = \delta$, and

$$\begin{aligned}
\epsilon_i(k-1) + \epsilon_{i+1}(k-1) + u_i &= u_i + \sum_{j=i+1}^{\delta-1} u_j \cdot \binom{k}{j-i} + \sum_{j=i}^{\delta-1} u_j \cdot \binom{k}{j-i+1} \\
&= \sum_{j=i}^{\delta-1} u_j \cdot \left[\binom{k}{j-i} + \binom{k}{j-i+1} \right] \\
&= \sum_{j=i}^{\delta-1} u_j \cdot \binom{k+1}{j-i+1} = \epsilon_i(k)
\end{aligned}$$

Computation of Consecutive Values of a Polynomial [4]

If ℓ denotes the length of the interval (the number of values), i.e. $0 \leq k < \ell$, then the error is bounded by:

$$\max_{0 \leq k < \ell} \epsilon_0(k) \leq \sum_{i=0}^{\delta-1} u_i \cdot \binom{\ell}{i+1} = \sum_{i=0}^{\delta-1} 2^{-n_i} \cdot \binom{\ell}{i+1}.$$

The values of n_i will be determined so that this error bound is less than some given error bound E. This can be done by determining n_0, then n_1, then n_2, and so on, each time by taking the smallest n_i (multiple of the word size) such that $2^{-n_i} \cdot \binom{\ell}{i+1}$ is less than a fraction of the remaining error bound.

Note: if $n_{i+1} = n_i$, one could have taken $\epsilon'_i(k) = \epsilon_{i+1}(k)$, but the gain would probably be low (since $\epsilon_{i+1}(k) \sim k \cdot u_{i+1} = k \cdot u_i$ at least) and the formulas would probably be much more complicated.

Error Bound on the Approximation of P_k by Q_k

On the interval J_k: $P_k(m) - Q_k(m) = \sum_{i=\delta+1}^{d} a_i(k) \cdot \binom{m}{i}$

with $a_i(k) = \Delta^i P(kT_J) = \sum_{j=i}^{d} a_j(0) \cdot \binom{kT_J}{j-i}$.

Thus

$$P_k(m) - Q_k(m) = \sum_{j=\delta+1}^{d} a_j(0) \sum_{i=\delta+1}^{j} \binom{kT_J}{j-i} \binom{m}{i}$$

with $0 \leq m \leq T_J - 1$ and $0 \leq kT_J \leq T_I - T_J$. This gives the following error bound on the approximation of P_k by Q_k:

$$\sum_{j=\delta+1}^{d} |a_j(0)| \sum_{i=\delta+1}^{j} \binom{T_I - T_J}{j-i} \binom{T_J - 1}{i}$$

but one can get better dynamical bounds by considering the sign of $a_j(0)$.

Summary for the Search for HR-Cases

1. Determine the exponent range of f; f will be scaled by a power of the radix, so that we are interested in the values modulo 1: $\tilde{f} = \beta^s \cdot f$.
2. Determine the admissible final error bound ε_0 on $\tilde{f}$.
3. Assume we have a degree-d polynomial approximation $P(m)$ to $\tilde{f}(x)$ with an error bound ε_f, where $x = x_0 + m \cdot d_x$ (the evaluation error of the polynomial is not taken into account here).
4. $P_k(X) = P(X + kT_J)$ will be approximated by degree-δ polynomials Q_k. The coefficients $a_i(k)$ of Q_k are in fact polynomials in k of degree $d - i$, where $0 \leq i \leq \delta$. The initial coefficients $b_{i,j}(0)$ of these polynomials $a_i(k)$ in the binomial base are computed with an error bounded by $2^{i+j}e$, where e is a bound on the evaluation error of the polynomial. For the ulp error condition, we want $2^{i+j}e \leq 2^{-n_j-1}$, thus $e \leq 2^{-i-j-n_j-1}$, where the values of the n_j's can be determined so that the error on $a_i(k)$ is less than some given bound:

$$\sum_{j=0}^{d-i-1} 2^{-n_j} \cdot \binom{T_I/T_J}{j+1} \leq E_i$$